AUGMENTED HILBERT SERIES OF NUMERICAL SEMIGROUPS

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Abstract

A numerical semigroup S is a subset of the non-negative integers containing 0 that is closed under addition. The Hilbert series of S (a formal power series equal to the sum of terms t^n over all $n \in S$) can be expressed as a rational function in t whose numerator is characterized in terms of the topology of a simplicial complex determined by membership in S. In this paper, we obtain analogous rational expressions for the related power series whose coefficient of t^n equals $f(n)$ for one of several semigroup-theoretic invariants $f : S \to \mathbb{R}$ known to be eventually quasipolynomial.

1. Introduction

A *numerical semigroup* is a subset $S \subset \mathbb{Z}_{\geq 0}$ containing 0 that is closed under addition and has finite complement, and a *factorization* of an element $n \in S$ is an expression of n as a sum of generators of S. A clear trend that has emerged in the study of numerical semigroups is the eventually quasipolynomial behavior of arithmetic invariants derived from their factorization structure [14]. More specifically, each of these invariants (which we call *S-invariants*) is a function assigning to each element $n \in S$ a value determined by the possible factorizations of n in S. This includes invariants from discrete optimization such as maximum and minimum factorization length [2], distinct factorization length count [11], and maximum and

minimum 0-norm [1], as well as more semigroup-theoretic invariants like the delta set [7], ω-primality [12], and the catenary degree [6], each of which agrees with a quasipolynomial for large input.

When (eventually) quasipolynomial functions arise in combinatorial settings, there are several potential ways to study them: (i) directly, using tools specific to the setting in question; (ii) via combinatorial commutative algebra, using Hilbert functions of graded modules; and (iii) via rational generating functions. Approaches (ii) and (iii) were largely pioneered by Stanley [17], among others, and carry with them powerful algebraic tools. The eventually quasipolynomial behavior of each semigroup invariant mentioned above was initially examined using standard semigroup-theoretic tools, and more recently an approach using Hilbert functions was developed [11]. The goal of this paper is to initiate the use of approach (iii) in studying S-invariants.

To date, rational generating functions have been used to study several aspects of numerical semigroups [4, 8], primarily using the *Hilbert series*

$$\mathcal{H}(S;t) = \sum_{n \in S} t^n = \frac{\mathcal{K}(S;t)}{(1-t^{n_1}) \cdots (1-t^{n_k})}$$

associated to each numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$. A natural consequence of the Hilbert syzygy theorem from commutative algebra [10] states that the numerator $\mathcal{K}(S;t)$ in the second expression above is a polynomial in t whose coefficients are obtained from the graded Betti numbers of the defining toric ideal of S. An alternative characterization of the coefficients of $\mathcal{K}(S;t)$ (stated formally in Theorem 2) uses the topology of a simplicial complex determined by membership in S [5]. One of the key selling points of the latter characterization is that it is given entirely in terms of the underlying semigroup S, without the theoretical overhead often necessary when incorporating commutative algebra techniques.

The primary goal of this paper is to obtain analogous rational expressions for various *augmented Hilbert series*, which we define to be series of the form

$$\mathcal{H}_f(S;t) = \sum_{n \in S} f(n) t^n$$

where f is some S-invariant admitting eventually quasipolynomial behavior. We give two such expressions: (i) when $f(n)$ counts the number of distinct factorization lengths of n (Proposition 1) and (ii) when $f(n)$ is the maximum or minimum factorization length of n (Theorem 3). Examples 3 and 4 illustrate the need for distinct rational forms for these invariants. We also specify how to obtain the *dissonance point* of each quasipolynomial function f (i.e., the optimal bound on the start of quasipolynomiality) from the numerator of its rational generating function (Theorem 4). Lastly, we examine these rational expressions under the operation of gluing numerical semigroups (Section 5) and give a closed form for each rational expression in the special case when S has 2 generators (Section 6).

2. Background

Definition 1. A *numerical semigroup* S is a cofinite, additive subsemigroup of $\mathbb{Z}_{\geq 0}$. When we write $S = \langle n_1, \ldots, n_k \rangle$ in terms of generators, we assume $n_1 < \cdots < n_k$. The *Frobenius number* of S is the largest integer $\mathsf{F}(S)$ lying in the complement of S. A *factorization* of $n \in S$ is an expression

$$n = a_1 n_1 + \cdots + a_k n_k$$

of n as a sum of generators of S, and the *length* of a factorization is the sum $a_1 + \cdots + a_k$. The *set of factorizations* of $n \in S$ is

$$\mathsf{Z}_S(n) = \{a \in \mathbb{Z}^k_{\geq 0} : n = a_1 n_1 + \cdots + a_k n_k\}$$

and the *length set* of n is the set

$$\mathsf{L}_S(n) = \{a_1 + \cdots + a_k : a \in \mathsf{Z}_S(n)\}$$

of all possible factorization lengths of n. The *maximum and minimum factorization length functions*, and the *length denumerant* function, are defined as

$$\mathsf{M}_S(n) = \max \mathsf{L}_S(n) \qquad \mathsf{m}_S(n) = \min \mathsf{L}_S(n) \qquad \text{and} \qquad \mathsf{l}_S(n) = |\mathsf{L}_S(n)|,$$

respectively. The *Apéry set* of an element $n \in S$ is the set

$$\operatorname{Ap}(S; n) = \{m \in S : m - n \notin S\}.$$

It can be easily shown that $|\operatorname{Ap}(S; n)| = n$ for any $n \in S$.

Notation 1. Unless otherwise stated, thoughout the paper, $S = \langle n_1, \ldots, n_k \rangle$ denotes a numerical semigroup with a fixed generating set $n_1 < \cdots < n_k$.

Definition 2. A function $f : \mathbb{Z} \to \mathbb{R}$ is an *S-invariant* if $f(n) = 0$ for all $n \notin S$.

Definition 3. A function $f : \mathbb{Z} \to \mathbb{R}$ is an *r-quasipolynomial* of degree α if

$$f(n) = a_\alpha(n) n^\alpha + \cdots + a_1(n) n + a_0(n)$$

for periodic functions $a_0, \ldots, a_\alpha$, whose periods all divide r, with a_α not identically 0. We say f is *eventually quasipolynomial* if the above equality holds for all $n \gg 0$.

Theorem 1 ([2, 11]). *For sufficiently large $n \in S$,*

$$\mathsf{M}_S(n + n_1) = \mathsf{M}_S(n) + 1, \qquad \mathsf{m}_S(n + n_k) = \mathsf{m}_S(n) + 1,$$
$$\text{and} \qquad \mathsf{l}_S(n + n_1 n_k) = \mathsf{l}_S(n) + \tfrac{1}{d}(n_k - n_1),$$

where $d = \gcd\{n_i - n_{i-1} : i = 2, \ldots, d\}$. *In particular, the* S*-invariants* M_S, m_S, *and* l_S *are each eventually quasilinear.*

Definition 4. The *Hilbert series* of S is the formal power series

$$\mathcal{H}(S;t) = \sum_{n \in S} t^n \in \mathbb{Z}[\![t]\!].$$

Given $n \in S$, the *squarefree divisor complex* Δ_n is a simplicial complex on the ground set $[k] = \{1, \ldots, k\}$ where $F \in \Delta_n$ if $n - n_F \in S$, where $n_F = \sum_{i \in F} n_i$. The *Euler characteristic* of a simplicial complex Δ is the alternating sum

$$\chi(\Delta) = \sum_{F \in \Delta_n} (-1)^{|F|}.$$

Remark 1. The definition of Euler characteristic above differs slightly from the usual topological definition, but has the advantage that $\chi(\Delta) = 0$ for any contractible simplicial complex Δ.

Theorem 2 ([5]). *The Hilbert series of* S *can be written as*

$$\mathcal{H}(S;t) = \sum_{n \in S} t^n = \frac{\sum_{a \in \mathrm{Ap}(S;n_1)} t^a}{1 - t^{n_1}} = \frac{\sum_{m \in S} \chi(\Delta_m) t^m}{(1 - t^{n_1}) \cdots (1 - t^{n_k})},$$

where both numerators have finitely many terms.

Example 1. For $S = \langle 6, 9, 20 \rangle$, Theorem 2 yields

$$\mathcal{H}(S;t) = \frac{1 + t^9 + t^{20} + t^{29} + t^{40} + t^{49}}{1 - t^6} = \frac{1 - t^{18} - t^{60} + t^{78}}{(1 - t^6)(1 - t^9)(1 - t^{20})}.$$

Here, $\mathrm{Ap}(S;6) = \{0, 49, 20, 9, 40, 29\}$ and each entry is distinct modulo 6. The elements

$$18 = 3 \cdot 6 = 2 \cdot 9 \qquad \text{and} \qquad 60 = 4 \cdot 6 + 4 \cdot 9 = 3 \cdot 20$$

are, respectively, the first element that can be factored using 6's and 9's and the first element that can be factored using 6's, 9's, and 20's. In particular, these two elements encode minimal relations between the generators of S, viewed as minimal "trades" from one factorization to another. Moreover,

$$78 = 7 \cdot 6 + 4 \cdot 9 = 2 \cdot 9 + 3 \cdot 20$$

is the first element in which two distinct sequences of trades between factorizations are possible: one can perform the exchange $3 \cdot 6 \rightsquigarrow 2 \cdot 9$ followed by $4 \cdot 6 + 4 \cdot 9 \rightsquigarrow 3 \cdot 20$, or these trades can be applied in the reverse order. This represents a "relation between minimal relations". These properties are encoded in the element's respective squarefree divisor complexes, since Δ_{18} and Δ_{60} are each disconnected, and Δ_{78} is connected but has nontrivial 1-dimensional homology.

Remark 2. One remarkable aspect of Theorem 2 is that simple algebraic manipulation of the rational expression of the Hilbert series reveals additional structural information about the underlying semigroup. Indeed, cancelling all common factors in Example 1 yields $\mathsf{P}_S(t)/(1-t)$ (see [8]), where

$$\begin{aligned}\mathsf{P}_S(t) = 1 - t \quad &+ t^6 \ - t^7 \ + t^9 \ - t^{10} + t^{12} - t^{13} + t^{15} - t^{16} + t^{18} \\ &- t^{19} + t^{20} - t^{22} + t^{24} - t^{25} + t^{26} - t^{28} + t^{29} - t^{31} + t^{32} \\ &- t^{34} + t^{35} - t^{37} + t^{38} - t^{43} + t^{44}\end{aligned}$$

has significantly more terms than the numerator of either form in Theorem 2. This is not a coincidence: since they represent the same power series, the fewer terms that appear in a particular expression, the more information each term must encode.

3. Numerators of Augmented Hilbert Series

In this section, we formally introduce augmented Hilbert series of a general semigroup invariant f (Definition 5), present two rational expressions in the spirit of Theorem 2 (Proposition 1 and Theorem 3), and illustrate and compare their use when f is one of the S-invariants appearing in Theorem 1 (Examples 3 and 4).

Definition 5. Fix an S-invariant f. The *augmented Hilbert series* of S with respect to f is the formal power series

$$\mathcal{H}_f(S;t) = \sum_{n\in S} f(n)t^n.$$

Given $n \in S$, the *weighted Euler characteristic of* Δ_n is defined as

$$\chi_f(\Delta_n) = \sum_{F\in\Delta_n} (-1)^{|F|} f(n - n_F),$$

and the *augmented Euler characteristic of* Δ_n is defined as

$$\widehat{\chi}_f(\Delta_n) = \sum_{F\in\Delta} (-1)^{|F|}(f(n - n_F) + |F|).$$

Example 2. Let $S = \langle 6, 9, 20\rangle$. The complex Δ_{138} is given in Figure 1, and each face F is labeled with the value $\mathsf{M}_S(138 - n_F)$. Together with $\mathsf{M}_S(138) = 23$ as the label for the empty face, we obtain

$$\chi_{\mathsf{M}_S}(\Delta_{138}) = \widehat{\chi}_{\mathsf{M}_S}(\Delta_{138}) = 0,$$

in part because the label of each face containing the vertex 6 matches its label on the face obtained by deleting 6.

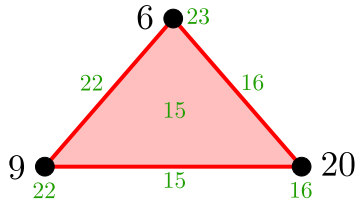

Figure 1: The complex Δ_{138} in $S = \langle 6, 9, 20\rangle$ in Example 2.

Proposition 1. *Fix an S-invariant f. For any fixed $p \in \mathbb{Z}_{\geq 1}$, we have*

$$\mathcal{H}_f(S;t) = \frac{\sum_{n\in S}(f(n) - 2f(n-p) + f(n-2p))t^n}{(1-t^p)^2} = \frac{\sum_{n\in S}\chi_f(\Delta_n)t^n}{\prod_{i=1}^{k}(1-t^{n_i})}.$$

Proof. Clearing respective denominators yields

$$\begin{aligned}(1-t^p)^2\sum_{n\in S} f(n)t^n &= \sum_{n\in S} f(n)t^n - \sum_{n\in S} 2f(n)t^{n+p} + \sum_{n\in S} f(n)t^{n+2p}\\ &= \sum_{n\in S}(f(n) - 2f(n-p) + f(n-2p))t^n,\end{aligned}$$

which proves the first equality, and the second equality follows from

$$\begin{aligned}\sum_{m\in S}\chi_f(\Delta_m)t^m &= \sum_{m\in S}\sum_{\substack{F\subseteq[k]\\ m-n_F\in S}}(-1)^{|F|}f(m-n_F)t^m = \sum_{A\subseteq[k]}\sum_{n\in S}(-1)^{|A|}f(n)t^{n+n_A}\\ &= \Big(\sum_{A\subseteq[k]}(-1)^{|A|}t^{n_A}\Big)\Big(\sum_{n\in S}f(n)t^n\Big) = \Big(\prod_{i=1}^{k}(1-t^{n_i})\Big)\Big(\sum_{n\in S}f(n)t^n\Big),\end{aligned}$$

where the second step uses the substitution $m = n + n_A$. □

Example 3. For $S = \langle 9, 10, 23\rangle$, we have

$$\begin{aligned}\sum_{n\in S}\chi_{\mathsf{M}_S}(n)t^n = t^9 + t^{10} + \ \ t^{18} + t^{20} + t^{23} + \ \ t^{27} + t^{30} + \ \ t^{36} + t^{40} + t^{45}\\ - t^{46} - 3t^{50} + t^{54} - t^{55} - \ \ t^{56} - t^{59} - 4t^{63} - t^{64} - t^{66}\\ - t^{68} + 2t^{73} - t^{76} - t^{77} + 3t^{86} - t^{90} + t^{113},\end{aligned}$$

whereas

$$\sum_{n\in S}\widehat{\chi}_{\mathsf{M}_S}(n)t^n = \ -2t^{46} - 4t^{50} - 5t^{63} + 5t^{73} + 6t^{86} - t^{90} + t^{113}.$$

This difference in number of terms occurs in nearly every example of $\mathcal{H}_{\mathsf{M}_S}(S;t)$ the authors have computed, and illustrates the primary reason for Theorem 3: filtering many of the extraneous terms from the first expression above. At play here is the philosophy discussed in Remark 2, namely that expressions with fewer terms necessarily encode more combinatorial information per term.

Example 4. For $S = \langle 9, 10, 23\rangle$ as in Example 3, the polynomials

$$\sum_{n\in S} \chi_{\mathsf{l}_S}(n)t^n = 1 - t^{140}$$

and

$$\begin{aligned}\sum_{n\in S} \widehat{\chi}_{\mathsf{l}_S}(n)t^n = 1 - \ & t^9 - t^{10} - \ t^{18} - t^{20} - t^{23} - \ t^{27} - \ t^{30} - t^{36} - t^{40} \\ & - t^{45} - t^{46} - \ t^{50} - t^{54} + t^{55} + \ t^{56} + \ t^{59} - t^{63} + t^{64} \\ & + t^{66} + t^{68} + 3t^{73} + t^{76} + t^{77} + 3t^{86} - t^{140}\end{aligned}$$

also differ greatly in the number of terms, but in the opposite direction. This is in part because $\mathsf{L}(0) = \{0\}$ for every numerical semigroup S, as the lack of a constant term in $\mathcal{H}_{\mathsf{l}_S}(S;t)$ adds many erroneous terms in the numerator of Proposition 1 that the constant term 1 in $\mathcal{H}_{\mathsf{l}_S}(S;t)$ avoids. Additionally, this example illustrates that examining S-invariants via generating functions will sometimes require specialized expressions, rather than a "one-size-fits-all" characterization.

Notation 2. In what follows, we make heavy use of the power series

$$z(t) = \prod_{i=1}^{k} \frac{1}{1-t^{n_i}} \qquad \text{and} \qquad \lambda(t) = \sum_{i=1}^{k} \frac{t^{n_i}}{1-t^{n_i}},$$

the second of which often occurs in the form $z(t)\lambda(t)$ (for instance, in Theorem 3). The coefficient of t^n in the series $z(t)$ (usually notated as $\mathcal{H}_\partial(S;t)$ in the literature) equals the number of factorizations of $n \in S$ (known as the *denumerant* of n), while the coefficients of $z(t)\lambda(t)$ are described in Lemma 1.

Lemma 1. *The power series $z(t)\lambda(t)$ is given by*

$$z(t)\lambda(t) = \sum_{n\in S} \ell(n)t^n,$$

where $\ell(n)$ denotes the sum of the lengths of every factorization of $n \in S$.

Proof. Rewrite $z(t)\lambda(t)$ as

$$z(t)\lambda(t) = \sum_{i=1}^{k} \frac{t^{n_i}}{(1-t^{n_i})^2}\Big(\prod_{j\neq i} \frac{1}{1-t^{n_j}}\Big) = \sum_{i=1}^{k} \Big(\sum_{a_i\geq 0} a_i t^{a_i n_i}\Big)\Big(\prod_{j\neq i}\Big(\sum_{a_j\geq 0} t^{a_j n_j}\Big)\Big).$$

In the final expression above, when expanding the product inside the outermost sum, the term t^n appears once for each factorization of n in S, with coefficient equal to the number of copies of n_i appearing in that factorization. As such,

$$z(t)\lambda(t) = \sum_{i=1}^{k}\sum_{n\in S}\sum_{a\in \mathsf{Z}(n)} a_i t^n \ = \sum_{n\in S}\sum_{a\in\mathsf{Z}(n)} |a|t^n = \sum_{n\in S} \ell(n)t^n,$$

as desired. □

Theorem 3. *Fix an S-invariant f. The augmented Hilbert series of f is given by*

$$\begin{aligned}\mathcal{H}_f(S;t) &= z(t)\lambda(t)\sum_{n\in S}\chi(\Delta_n)t^n + z(t)\sum_{n\in S}\widehat{\chi}_f(\Delta_n)t^n \\ &= \lambda(t)\mathcal{H}(S;t) + \frac{\sum_{n\in S}\widehat{\chi}_f(\Delta_n)t^n}{(1-t^{n_1})\cdots(1-t^{n_k})}.\end{aligned}$$

Proof. Multiplying both sides by the denominator of $z(t)$, Proposition 1 implies

$$\left(\prod_{i=1}^{k}(1-t^{n_i})\right)\sum_{n\in S}f(n)t^n = \sum_{m\in S}\sum_{F\in\Delta_m}(-1)^{|F|}f(m-n_F)t^m.$$

In the second term on the right-hand side of the claimed equality, we have

$$\begin{aligned}\sum_{m\in S}\widehat{\chi}_f(\Delta_m)t^m &= \sum_{m\in S}\sum_{F\in\Delta_m}(-1)^{|F|}(f(m-n_F)+|F|)t^m \\ &= \sum_{m\in S}\sum_{F\in\Delta_m}(-1)^{|F|}f(m-n_F)t^m + \sum_{m\in S}\sum_{F\in\Delta_m}(-1)^{|F|}|F|t^m,\end{aligned}$$

so it suffices to show that

$$\left(\sum_{i=1}^{k}\frac{t^{n_i}}{1-t^{n_i}}\right)\sum_{m\in S}\sum_{G\in\Delta_m}(-1)^{|G|}t^m + \sum_{m\in S}\sum_{F\in\Delta_m}(-1)^{|F|}|F|t^m = 0.$$

Indeed, multiplying the first part by $\prod_{j=1}^{k}(1-t^{n_j})$ yields

$$\begin{aligned}\sum_{i=1}^{k}t^{n_i}\left(\prod_{j\neq i}^{k}(1-t^{n_j})\right)\sum_{m\in S}\sum_{G\in\Delta_m}(-1)^{|G|}t^m &= \sum_{m\in S}\sum_{i=1}^{k}\sum_{\substack{A\subseteq[k]\\ i\in A}}(-1)^{|A|-1}\sum_{G\in\Delta_m}(-1)^{|G|}t^{m+n_A} \\ &= \sum_{m\in S}\sum_{A\subseteq[k]}(-1)^{|A|-1}|A|\sum_{G\in\Delta_m}(-1)^{|G|}t^{m+n_A} \\ &= \sum_{m\in S}\sum_{F\in\Delta_m}(-1)^{|F|-1}|F|\sum_{G\in\Delta_m}(-1)^{|G|}t^m\end{aligned}$$

and multiplying the second part by the same factor yields

$$\begin{aligned}\left(\prod_{j=1}^{k}(1-t^{n_j})\right)\sum_{m\in S}\sum_{F\in\Delta_m}(-1)^{|F|}|F|t^m &= \sum_{m\in S}\sum_{A\subseteq[k]}(-1)^{|A|}\sum_{F\in\Delta_m}(-1)^{|F|}t^{m+n_A} \\ &= \sum_{m\in S}\sum_{G\in\Delta_m}(-1)^{|G|}\sum_{F\in\Delta_m}(-1)^{|F|}t^m\end{aligned}$$

which completes the proof. □

4. The Dissonance Point

The numerator of each rational expression in Proposition 1 and Theorem 3 has finite degree when f is any S-invariant listed in Theorem 1. This follows from Theorem 1 and general facts from the theory of generating function [18], but we prove this fact in Proposition 2 using weighted and augmented Euler characteristics, as a demonstration of their utility.

The other main result of this section is Theorem 4, which demonstrates that when $f = \mathsf{M}_S$ or $f = \mathsf{m}_S$, we can recover from the degree of $\sum_{n\in S} \widehat{\chi}_f(\Delta_n)t^n$ the minimum integer input after which f becomes truly quasipolynomial. Note that by Proposition 1, this fact is immediate if the coefficients $\chi_f(\Delta_n)$ are used in place of $\widehat{\chi}_f(\Delta_n)$ for *any* eventually quasipolynomial function f.

Definition 6. Fix an S-invariant f that agrees with a quasipolynomial function $g : \mathbb{Z} \to \mathbb{R}$ for sufficiently large input values. The *dissonance point of* f is the largest integer $n \geq \mathsf{F}(S)$ such that $f(n) \neq g(n)$. We say the semigroup S is f*-harmonic* if $f(n) = g(n)$ for every $n \in S$.

Example 5. Let $S = \langle 9, 10, 23\rangle$ from Example 3. The dissonance point of M_S is 71, since $\mathsf{Z}(71) = \{(2, 3, 1)\}$ but $\mathsf{Z}(80) = \{(3, 3, 1), (0, 8, 0)\}$, so

$$8 = \mathsf{M}_S(80) > \mathsf{M}_S(71) + 1 = 7.$$

In particular, the longest factorization of 80 does not have any copies of the first generator. Generally, longer factorizations will involve more small generators than large generators, but even though $(3, 3, 1)$ has more copies of the smallest generator, it has enough larger generators to afford $(0, 8, 0)$ higher efficiency. This is exacerbated by the fact that 9 and 10 are close together, while 23 is significantly larger than both.

On the other hand, $S = \langle 6, 9, 20\rangle$ is M_S-harmonic, since

$$\mathsf{M}_S(n + 6) = \mathsf{M}_S(n) + 1$$

for every $n \in S$ by Theorem 1 and exhaustive computation for small n using, for instance, the `GAP` package `numericalsgps` [9].

Proposition 2. *If f is one of the S-invariants appearing in Theorem 1, then $\sum_{n\geq 0} \chi_f(\Delta_n)t^n$ and $\sum_{n\geq 0} \widehat{\chi}_f(\Delta_n)t^n$ have finitely many terms.*

Proof. We must show

$$\chi_f(\Delta_n) = \widehat{\chi}_f(\Delta_n) = 0$$

for all sufficiently large n. If $f = \mathsf{m}_S$, then f satisfies $f(n + n_k) = f(n) + 1$ for

sufficiently large n, so provided that $n > \mathsf{F}(S) + n_{[k]}$ also holds, we have

$$\begin{aligned}\chi_f(\Delta_n) &= \sum_{F\subseteq[k]} (-1)^{|F|} f(n-n_F) \\ &= \sum_{F\subseteq[k-1]} (-1)^{|F|} f(n-n_F) + \sum_{F\subseteq[k-1]} (-1)^{|F|+1} f(n-n_F-n_k) \\ &= \sum_{F\subseteq[k-1]} (-1)^{|F|} (f(n-n_F) - f(n-n_F-n_k)) = \sum_{F\subseteq[k-1]} (-1)^{|F|} = 0.\end{aligned}$$

Additionally,

$$\widehat{\chi}_f(\Delta_n) - \chi_f(\Delta_n) = \sum_{F\subseteq[k]} (-1)^{|F|}|F| = 0,$$

which proves $\widehat{\chi}_f(\Delta_n) = 0$. Replacing n_k with n_1 throughout the above argument proves the same equalities hold for $f = \mathsf{M}_S$, leaving only the case $f = \mathsf{l}_S$. By Theorem 1, we have $f(n) = \frac{1}{d}(\mathsf{M}(n) - \mathsf{m}(n)) - l_0(n)$ for large n, where l_0 is some n_1n_k-periodic function. As such,

$$\sum_{n\in S} f(n)t^n = \frac{1}{d}\bigg(\sum_{n\in S} \mathsf{M}(n)t^n - \sum_{n\in S} \mathsf{m}(n)t^n\bigg) - \sum_{n\in S} l_0(n)t^n,$$

and by Proposition 1, each power series on the right-hand side is rational with denominator dividing $\prod_{i=1}^{k}(1-t^{n_i})$. This proves $\chi_f(\Delta_n) = 0$ for large n. Just as above, $\widehat{\chi}_f(\Delta_n) = 0$ then readily follows for large n, so the proof is complete. □

Theorem 4. *If $f = \mathsf{M}_S$ or $f = \mathsf{m}_S$, then the dissonance point of f is $d - n_{[k]}$, where*

$$d = \deg\bigg(\sum_{n\geq 0} \widehat{\chi}_f(\Delta_n)t^n\bigg).$$

Proof. Suppose $f = \mathsf{m}_S$, and let $m \in S$ denote the largest element of S such that $\mathsf{m}(m-n_k)+1 \neq \mathsf{m}(m)$. Clearly $m \leq d - n_{[k]}$, since each $n > m + n_{[k]}$ must have $\widehat{\chi}_f(\Delta_n) = 0$ by the proof of Proposition 2. Moreover,

$$\widehat{\chi}_f(\Delta_d) = \sum_{F\subseteq[k]} (-1)^{|F|}\big(f(d-n_F) + |F|\big) = (-1)^k\big(1 + f(d-n_{[k]}) - f(d-n_{[k-1]})\big)$$

is nonzero, proving the claim when $f = \mathsf{m}_S$. The case $f = \mathsf{M}_S$ follows analogously. □

5. Augmented Hilbert Series of Gluings

Gluing (Definition 7) is a method of combining two numerical semigroups S_1 and S_2 to obtain a numerical semigroup $S = d_1S_1 + d_2S_2$ whose factorization structure can

be expressed explicitly in terms of the factorizations of S_1 and S_2 [16]. Several families of numerical semigroups of interest in the literature (e.g., complete intersection, supersymmetric, telescopic) are described in terms of gluings. Moreover, the Hilbert series of S can be concisely expressed as

$$\mathcal{H}(S;t) = (1 - t^{d_1 d_2})\mathcal{H}(S_1; t^{d_1})\mathcal{H}(S_2; t^{d_2})$$

in terms of the Hilbert series of S_1 and S_2 (see [8]).

One might hope that a similar relation can be obtained for augmented Hilbert series, but unfortunately this is not the case. In fact, even gluing two harmonic numerical semigroups need not yield a harmonic numerical semigroup; see Example 6. However, if the gluing is sufficiently well-behaved (see Definition 8), then an expression for the augmented Hilbert series of S can be obtained (Theorem 5).

Remark 3. All results and definitions in this section are stated in terms of the maximum factorization length S-invariant M_S, but analogous results (with analogous proofs) also hold for the minimum factorization length S-invariant m_S.

Definition 7. Fix numerical semigroups S_1 and S_2, and elements $d_1 \in S_2$ and $d_2 \in S_1$ that are not minimal generators of their respective semigroups. We say $S = d_1 S_1 + d_2 S_2$ is a *gluing of* S_1 *and* S_2 if $\gcd(d_1, d_2) = 1$.

Example 6. Let $S_1 = \langle 6, 10, 15\rangle$ and $S_2 = \langle 5, 7\rangle$, and let

$$S = 23S_1 + 27S_2 = \langle 138, 230, 345, 135, 162\rangle.$$

Both S_1 and S_2 are M_S-harmonic (and supersymmetric, one of the most well-behaved families of numerical semigroups under gluing), but the glued numerical semigroup S fails to satisfy $\mathsf{M}_S(n + n_1) = \mathsf{M}_S(n) + 1$ for each n in the set

$$\left\{ \begin{array}{c} 831,\ \ 969,\ \ 993, 1061, 1131, 1155, 1199, 1223, 1291, 1293, \\ 1317, 1361, 1385, 1429, 1453, 1455, 1479, 1523, 1547, 1591, \\ 1615, 1617, 1685, 1709, 1753, 1777, 1847, 1915, 1939,\ \ 2077 \end{array} \right\}$$

(this can be verified using the `GAP` package `numericalsgps` [9]). The primary issue is that the images of the smallest generators of S_1 and S_2 are relatively close in S, a property that was observed by the second author when writing [2] to correlate with a large dissonance point for maximum factorization length.

Definition 8. Resume notation from Definition 7. We say S is a M_S*-harmonic gluing* if every $n \in S$ satifies $\mathsf{M}_S(n) = \mathsf{M}_{S_1}(n') + \mathsf{M}_{S_2}(n'')$, where $n = d_1 n' + d_2 n''$ for $n' \in S_1$, $n'' \in S_2$, and n' maximal among all such expressions. Note that this property is dependent on the order of S_1 and S_2. We define an m_S*-harmonic gluing* analogously, where the expression $n = d_1 n' + d_2 n''$ is chosen so that n'' is maximal.

Theorem 5. *If $S = d_1S_1 + d_2S_2$ is an M_S-harmonic gluing, then*

$$\mathcal{H}_{\mathsf{M}_S}(S;t) = \mathcal{H}(S_1;t^{d_1})\Big(\sum_{n\in A_2}\mathsf{M}_{S_2}(n)(t^{d_2})^n\Big) + \mathcal{H}_{\mathsf{M}_{S_1}}(S_1;t^{d_1})\Big(\sum_{n\in A_2}(t^{d_2})^n\Big),$$

where $A_2 = \mathrm{Ap}(S_2;d_1)$, and if S is an m_S-harmonic gluing, then

$$\mathcal{H}_{\mathsf{m}_S}(S;t) = \Big(\sum_{n\in A_1}\mathsf{m}_{S_1}(n)(t^{d_1})^n\Big)\mathcal{H}(S_2;t^{d_2}) + \Big(\sum_{n\in A_1}(t^{d_1})^n\Big)\mathcal{H}_{\mathsf{m}_{S_2}}(S_2;t^{d_2}),$$

where $A_1 = \mathrm{Ap}(S_1;d_2)$

Proof. The key is that whenever $n = d_1n' + d_2n'' \in S$ with $n' \in S_1$ and $n'' \in S_2$, we have n' maximal among all such expressions for n if and only if $n'' \in \mathrm{Ap}(S_2;d_1)$. Indeed, if $n'' - d_1 \in S_2$, then we can write $n = d_1(n' + d_2) + d_2(n'' - d_1)$, and the converse holds since $\gcd(d_1,d_2) = 1$. This implies the coefficient of t^n obtained from expanding the right-hand side of the first equality is $\mathsf{M}_{S_1}(n') + \mathsf{M}_{S_2}(n'')$, so the harmonic assumption on S proves the first equality. An analogous argument proves the second equality. □

6. Numerical Semigroups With 2 Generators

In this section, we restrict our attention to the case $S = \langle n_1, n_2\rangle$.

Theorem 6. *If $S = \langle n_1, n_2\rangle$, then*

$$\sum_{n\in S}\widehat{\chi}_{\mathsf{M}_S}(\Delta_n)t^n = -n_1t^{n_1n_2} \qquad \textit{and} \qquad \sum_{n\in S}\widehat{\chi}_{\mathsf{m}_S}(\Delta_n)t^n = -n_2t^{n_1n_2}.$$

Proof. It suffices to prove the first equality, as the second follows analogously. We use the well-known fact that

$$\mathrm{Ap}(S;n_1) = \{0, n_2, \ldots, (n_1-1)n_2\},$$

every element of which is uniquely factorable, and that $\mathsf{M}_S(n + n_1) = \mathsf{M}_S(n) + 1$ for every $n \in S$ [16]. As such,

$$\begin{aligned}\widehat{\chi}_{\mathsf{M}_S}(\Delta_{n_1n_2}) &= \mathsf{M}_S(n_1n_2) - (\mathsf{M}_S(n_1n_2 - n_1) + 1) - (\mathsf{M}_S(n_1n_2 - n_2) + 1)\\ &= n_2 - (n_2 - 1 + 1) - (n_1 - 1 + 1) = -n_1.\end{aligned}$$

For all other elements $n \neq n_1n_2$, the complex Δ_n is either (i) the a single vertex 1, (ii) the single vertex 2, or (iii) the full simplex $2^{[2]}$. In each case, one readily checks that $\widehat{\chi}_{\mathsf{M}_S}(\Delta_n) = 0$, thereby completing the proof. □

Remark 4. It is known that for $S = \langle n_1, n_2 \rangle$, no two factorizations of a given element $n \in S$ have the same length, so

$$\sum_{n \in S} \mathsf{l}_S(n) t^n = \sum_{n \in S} |Z_S(n)| t^n = z(t) = \frac{1}{(1 - t^{n_1})(1 - t^{n_2})}$$

Remark 5. The disparity between $\chi_{\mathsf{M}_S}(\Delta_n)$ and $\widehat{\chi}_{\mathsf{M}_S}(\Delta)$ is perhaps most exemplified in the case $S = \langle n_1, n_2 \rangle$. Indeed, for $S = \langle 9, 11 \rangle$, we have $\sum_{n \in S} \widehat{\chi}_{\mathsf{M}_S}(\Delta_n) t^n = -9t^{99}$ by Theorem 6, whereas

$$\begin{aligned} \sum_{n \in S} \chi_{\mathsf{M}_S}(\Delta_n) t^n = t^9 + t^{11} + t^{18} + t^{22} + t^{27} + t^{33} + t^{36} + t^{44} + t^{45} + t^{54} \\ + t^{55} + t^{63} + t^{66} + t^{72} + t^{77} + t^{81} + t^{88} + t^{90} - 7t^{99} \end{aligned}$$

has one additional term for each element of $\mathrm{Ap}(S; n_1)$ and $\mathrm{Ap}(S; n_2)$.

7. Future Work

The ω-primality invariant ω_S, a semigroup-theoretic measure of nonunique factorization [13], is also known to be eventually quasilinear over numerical semigroups. More precisely, for all sufficiently large $n \in S$,

$$\omega(n + n_1) = \omega(n) + 1.$$

Additionally, it is known [3] that the domain of ω_S can be naturally extended to the quotient group $\mathbb{Z}$, i.e., $\omega_S : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$, in such a way that sufficiently negative input values yield 0. In many cases, after the domain is extended in this way, the lower bound on n after which quasilinearity holds for ω_S can be significantly lowered.

Problem 1. Find rational expressions for the power series $\sum_{n \in S} \omega_S(n) t^n$ and its extension $\sum_{n \in \mathbb{Z}} \omega_S(n) t^n$ in the style of Proposition 1 or Theorem 3.

There are eventually quasipolynomial S-invariants that arise naturally in studying numerical semigroups whose period does not divide the product $n_1 \cdots n_k$. For example, writing $\ell^\infty(a)$ for the component-wise maximum of $a \in \mathbb{Z}_{\geq 0}^k$, it is not hard to show

$$n \mapsto \min\{\ell^\infty(a) : a \in \mathsf{Z}(n)\}$$

is eventually quasilinear in n with period dividing $n_1 + \cdots + n_k$. As this often does not divide the product $n_1 \cdots n_k$, the rational expressions in Proposition 1 and Theorem 3 will not have numerators with finite degree.

Problem 2. Develop an analogue of Proposition 1 and Theorem 3 for S-invariants whose periods do not divide the product $n_1 \cdots n_k$.

Given $n \in S$, define the simplicial complex ∇_n with vertex set $\mathsf{Z}(n)$ where $F \subset \mathsf{Z}(n)$ is a face of ∇_n whenever there is some generator appearing in every factorization in F. The complex ∇_n is topologically equivalent to Δ_n (this was first observed in [15]), and thus is sometimes used in place of Δ_n when examining Hilbert series of numerical semigroups via Theorem 2.

Problem 3. Find labelings of the simplicial complex ∇_n so that the weighted and augmented Euler characteristic matches those of Δ_n.

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